MARKOV PROCESSES, FMS180/MAS204, SPRING SEMESTER 2007
EXERCISES FOR MARKOV PROCESSES

Conditional probabilities

001. An office has 110 employees, of which 50 are women. Through a questionnaire from the canteen one has found how many are vegetarians:

	vegetarians	non-vegetarians
men	12	48
women	18	32

Consider an employee chosen at random.

(a) Compute the probability that the person is a vegetarian.

(b) Assume that the employee is a woman. What is the probability that she is a vegetarian?

(c) Are the events 'woman chosen' and 'vegetarian chosen' independent? Motivate your answer!

Parts (a) and (b) may be solved by picking suitable figures out of the table. Do that, but do also solve these problems in a more formal fashion by formulating conditional probabilities!

002. Consider a medical test that diagnoses a certain rare disease. It gives a positive answer (i.e., indicates the disease) with probability 0.98 if the patient does have the disease (this is called the *sensitivity* of the test), and with probability 0.04 if the patient does not have the disease (one minus this number is called the *specificity* of the test). It is known that in the whole population, 1 out of 1000 has the disease.

Compute the probability that a patient actually has the disease if the test outcome is positive (and the patient is chosen at random otherwise).

003. A call centre receives a random number of calls per day, with a Poisson distribution with mean m. A given call concerns a particular newly released product with probability p, independently of all other calls and of the total number of calls.

Compute the distribution (e.g. probability function) of the number of calls on a given day that concern the new product.

004. Let A and B be events. Let also $H_1, H_2, \ldots, H_n$ be events such that (i) $\cup_{i=1}^n H_i = \Omega$ (these events cover the whole sample space) and (ii) $H_i \cap H_j = \emptyset$ for $i \neq j$ (these events are mutually disjoint). Prove that it then holds that

$$P(A \mid B) = \sum_{i=1}^{n} P(A \mid H_i B) P(H_i \mid B),$$

provided all conditional probabilities are well-defined.

Modelling with Markov chains and processes

101. Anna, Beth and Christine play with a ball. Anna passes the ball on to Beth with probability 0.3 and to Christine with probability 0.7. Beth passes it on to Anna with probability 0.6 and to Christine with probability 0.4. Christine, finally, chooses between her friends with equal probabilities. All throws are carried out independently. The sequence 'who has the ball' is a Markov chain. Motivate this, define a suitable state space and write down the transition probability matrix.

102. On a table there is a dictionary in three volumes, A, B and C, in a pile. A stream of people uses the dictionary, independently of each other, in the following way: each person chooses one of the three volumes with equal probabilities, uses it, and puts it back on top of the pile. The pile can be ordered in six different ways:

ABC, ACB, BAC, BCA, CAB, CBA

(the left-most letter corresponds to the volume on top of the pile). The ways the books are ordered, after each access, in then a Markov chain. Motivate this and write down the transition probability matrix.

103. Cars, lorries and buses drive on a motorway, in one of the directions. It has been observed that after one out of ten cars follows a lorry, and after one out of ten cars follows a bus. After two out of three lorries follows another lorry, and after one out of three lorries follows a bus. After eight out of ten buses follows a car, and after one out of ten buses follows another bus.

 (a) Suppose that the stream of cars, buses and lorries forms a Markov chain. Draw the model graph and/or write down the transition probability matrix for this chain.

 (b) In practice, lorries often drive together in groups. Suppose instead that lorries always drive in groups of three, and that no cars or buses can 'sneak in' between lorries. After such a group there is always a bus. Keeping the Markov assumption except for the groups of lorries, draw the model graph and/or write down the transition probability matrix for this new chain.

104. A sequence of electrical impulses passes a measurement instrument that stores the largest value measured so far. Assume that the impulses at time points $1, 2, 3, \ldots$ can be modelled as independent random variables $Y_1, Y_2, Y_3, \ldots$ with a uniform distribution on $\{1, 2, 3, 4, 5\}$. Thus, if $X_1, X_2, X_3, \ldots$ are the values stored at time points $1, 2, 3, \ldots$, then

$$X_n = \max(Y_1, Y_2, \ldots, Y_n) \quad \text{for } n = 1, 2, 3, \ldots.$$

Motivate that $\{X_n\}_{n=1}^{\infty}$ is a Markov chain and write down the transition probability matrix.

105. *The self-fertilising Markov chain.* Some plants such as rice and wheat have both male and female organs. These plants reproduce through self-fertilisation. Consider a particular genetic locus at which either of the two alleles may be *A* or *a*. The possible genotypes are thus *AA*, *Aa* and *aa*. Upon reproduction, each of the two new alleles is chosen at random from the two present ones. Let X_n be the genotype at generation n (X_n is thus *AA*, *Aa* or *aa*), motivate that $\{X_n\}$ is a Markov chain and write down its transition probability matrix and/or model graph.

106. *The Wright-Fisher model.* This model originated in genetics, but we will here present it in a different framework.

Consider a population of N individuals, each of which may agree ('yes') or disagree ('no') with a certain statement. Time is discrete. Let X_n denote the number of 'yes' voters at time n. The dynamics of the system is as follows. From time n to time $n + 1$, each individual adopts the opinion 'yes' with probability $\theta_n = X_n/N$—the fraction of 'yes' voters in the population at time n—and consequently the opinion 'no' with probability $1 - \theta_n = (N - X_n)/N$. In other words, each individual is affected by the current opinion. However, at time $n + 1$ all individuals are assumed to make their decisions independently of each other.

Motivate that $\{X_n\}$ is a Markov chain and write down its transition probabilities.

Hint: given X_n, what is the conditional distribution of X_{n+1}?

107. In a radio show quiz, listeners may call in to try their skills. Each caller is given one question. If the answer is wrong, he/she wins nothing and a CD, which he/she could have won, is put 'on hold' (Sw: *i potten*). If the answer is correct, he/she wins the CD, plus all CDs that are currently 'on hold'. Each caller gives a correct answer with probability p, independently for all callers.

Let X_k be number of CDs 'on hold' just before caller number k gives his/her answer; then $\{X_k\}$ is a Markov chain. Give the state space of this chain, and draw its model graph and/or write down its transition probability matrix.

108. *Binary Markov source through binary symmetric channel.* A binary symmetric channel (BSC) is a communication channel over which 0's and 1's can be sent. A bit may be distorted during the transmission, that is, a 0 may be received as a 1 as vice versa. Each bit is distorted with probability ε, independently of other bits.

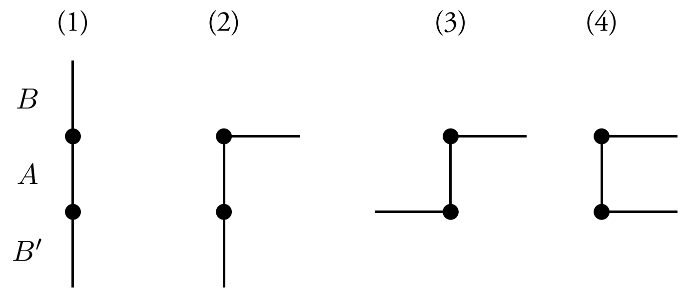
    Consider a Markov chain with state space $\{0, 1\}$, and let the sequence of symbols generated by this chain be sent over a BSC. Is the sequence received at the other end of the channel a Markov chain? Motivate your answer! You may assume that the Markov chain starts in state 0.

109. A certain kind of bacteria reproduce in the way that each individual splits into two with a constant intensity λ.

    (a) The waiting time until an individual splits is random. What is its distribution?
    (b) Let $X(t)$ be the number of bacteria at time t. Motivate that $\{X(t)\}_{t\geq 0}$ is a Markov process on the state space $\{1, 2, 3, \ldots\}$. Draw its model graph and write down the transition intensities.
    (c) Assuming that each bacterium dies with a constant intensity μ (hence, it may die before it splits), how would you modify the model graph (and state space) to accommodate this?

110. The following is a very simplified model of how proteins fold and unfold. Consider three sticks A, B and B' such that B and B' are connected to the two ends of A, which thus is in the middle. The end parts B and B' can both turn 90 degrees, in either direction, relative to A. This mechanism gives rise to four different geometrical configuration 1–4, shown below. Note that we are only interested in the geometry of the configurations, so that a configuration and its mirror images (horizontally, vertically or both) are the same state. In state 4, the end points of B and B' are as close as possible, so that, for example, a chemical reaction may occur.

    Assume that each end part B and B' turns 90 degrees at constant rate λ. If the turn can be made in either direction, they are chosen with equal probabilities. Under these assumptions the process 'configuration at time t' is a Markov process on the state space $\{1, 2, 3, 4\}$. Draw its model graph and write down its transition intensities.



111. A machine in a production plant has two units: cutting unit and computer unit. They fail with intensities 0.02 h^{-1} and 0.01 h^{-1}, respectively. When either of the units is broken, it is repaired by a repairman. The repair intensity for the cutting unit is 1 h^{-1} and for the computer unit it is 0.4 h^{-1}. When either unit has failed the system is halted and the other unit cannot fail.

    Draw a model graph of a Markov process that described the state of the system and indicate all intensities.

112. A fault tolerant computer system has two processors working in active redundance and one memory unit. The system is up if at least one processor and the memory unit is working, otherwise down. Each processor fails with constant intensity λ_P and the memory unit fails with intensity λ_M. When the

system is down it is shut off and no further components can fail. There is one repair man, who can repair a broken processor or memory unit at constant rate μ. When one processor and the memory unit have failed the repair man always works on the memory unit, in order to make the system working again as soon as possible.

(a) The system can be modelled by a Markov process. Draw the model graph write down the transition intensities.

(b) Now assume that the repair intensities for a processor and the memory unit are different, μ_P and μ_M say. Modify the model graph and transition intensities according to this scenario.

(c) As (b), but when a component of the system fails, the repair man starts working with that component and does not stop until finished, so that other components that may have failed in the mean time are taken care of afterwards. Hence, if a processor fails and then the memory unit, the memory unit is not given attention until the processor repair is finished.

113. Consider the following queueing system. Customers arrive to the system at constant intensity 0.5 s^{-1} (that is, as a Poisson process). Every customer needs service from a server. Service times are random, but with a common distribution which is exponential with mean 3 s. If the server is busy (with another customer) at an arrival, the newly arrived customer must wait in a queue. Customers are served in the order they arrive, so each arriving customer must line up at the end of the queue.

(a) The service intensity is constant. Why? What is its value?

(b) Let $X(t)$ be the number of customers in the system at time t, including the one currently being served. Then $\{X(t)\}_{t\geq 0}$ is a Markov process on the state space $\{0, 1, 2, \ldots\}$. Draw the model graph for this Markov process and write down the transition intensities.

(c) Now assuming that the space for waiting customers is limited, so that only 4 customers are allowed to wait (thus not counting the one being served), and assuming that customers arriving to a full system are blocked (they cannot enter), how would you modify the model graph?

(d) Go back to (b) but assume that there are two identical servers, so that up to two customers may be served at the same time. How would you modify the model graph?

114. Customers arrive to a service system according to a Poisson process with intensity 2. The system has two servers, and service times have an exponential distribution with mean $1/2$. Service times are independent of each other and of the arrivals. There is one waiting space for customers that arrive when both servers are busy. There may thus be a maximum of three customers in the system; two being served and one waiting. Customers that arrive when the one waiting space is occupied are rejected.

Let $X(t)$ be the number of customers in the system at time t, including those in service, if any. Then $\{X(t)\}$ is a discrete Markov process. Draw the model graph of this process, comprising all states and all transitions and their intensities.

115. Customers arrive to a service system according to a Poisson process with intensity λ. The system has one server, and each customer requires a random service time from an exponential distribution with mean $1/\mu$ ($\mu > \lambda$). The service times are independent of each other and of the arrivals. When the system becomes empty (no customers left) the served is shut down, and is not restarted until there are two customers in the system. There is an infinite waiting space.

Draw a model graph of a Markov process that describes the state of the system and indicate all intensities.

116. Consider a queueing system to which customers arrive according to a Poisson process with intensity λ. The system has one server, and each customer requires a service time from an exponential distribution with mean $1/\mu$. It holds that $\lambda < \mu$, and service times are independent of each other and of the arrival process.

The serves can either be turned *on* or *off*. Each time the system becomes empty, the server is turned off for maintenance for a time that has an exponential distribution with mean $1/\gamma$; then the server is turned on again. Notice that the server may be turned on without serving a customer.

When the server is turned off all arrivals are blocked; customers cannot join the queue and do not return. When the server is turned on, an arriving customer is either given service immediately (if the server is idle), or joins the queue (if the server is busy); the waiting space is unlimited.

Draw a model graph of a Markov process that describes the state of the system and indicate all intensities.

117. Consider a queueing system to which customers arrive as a Poisson process with intensity λ. The system has two servers, A and B. Each customer needs service from one of the servers, but which one is unimportant. Service times are exponentially distributed, with means $1/\mu_A$ and $1/\mu_B$ for servers A and B respectively. Here $1/\mu_A < 1/\mu_B$, so server A is the faster one. If an arriving customer finds both servers empty, he/she will therefore choose server A. If an arriving customer finds both servers busy, he/she will join a queue, waiting for service. This queue is common to both servers, and the number of waiting spaces is infinite. When a server becomes available and there are customers in the queue, the customer first in line occupies the available server. Customers do no interrupt their service to change server. Service times are independent of each other and of the arrival process.

    Draw a model graph of a Markov process that describes the system. Indicate all transition intensities.

118. Consider a service system to which customers arrive according to a Poisson process with intensity λ. The system has one server, and service times are independent (of each other and of the arrivals) with an exponential distribution with mean $1/\mu$. Customers that arrive when the server is busy join a queue, waiting for service. There is an infinite number of waiting spaces. Each time the system becomes empty, the server is shut down, for maintenance, for a time that has an exponential distribution with mean $1/\delta$. Any customers arriving during such maintenance periods join the queue.

    Define the states (k, U): the server is up (working) and there are k customers in the system (including the one possibly being served), and (k, D): the server is down and there are k customers in the system.

    Draw a model graph of the system, containing all states, transitions and transition intensities.

119. A machine in a production plant assembles two types of parts, A and B, into a product (a product thus consists of one part of type A and one part of type B). Parts of types A and B arrive at the machine according to two independent Poisson processes of intensities λ_A and λ_B, respectively. As soon as there is at least one part of each type available at the machine, they are assembled into a product and leave the machine. The time needed to do the assembly is approximated by zero. If parts arrive at the machine but cannot be assembled (because there happens to be no parts of the other type available), these parts are put in a storage but used as soon as parts of the other type arrive. This storage has room for M parts; parts that should be stored when the storage is full are lost (or you may think of the delivery process of these parts as being halted when the storage is full).

    Let $X(t) = (i, j)$ if there are i parts of type A and j parts of type B in the storage at time t. Then $\{X(t)\}$ is a discrete Markov process. Draw its model graph.

120. A storage is a production plant is modelled in the following way. The storage can hold a maximum of 4 units. Units are taken out of the storage, to be used in the production, according to a Poisson process with intensity 1 day^{-1}. When there is only 1 unit left in the storage, a delivery order for new units is placed. This delivery arrives after an independent exponentially distributed random time with mean 1/2 day. The storage may be emptied before the delivery arrives; the delivery order is then sustained. At the delivery the storage is always filled to the maximal 4 units.

    The number of units in the storage can be described by a discrete Markov process. Introduce suitable states and draw a model graph for this process; indicate all transition intensities.

Sample paths, Chapman-Kolmogorov equations etc.

201. In a simple model of the weather, each day is classified as 'sunny' or 'cloudy' and the sequence of weathers is modelled as a Markov chain with transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}.$$

Here the first row and column correspond to state 'sunny'.

(a) What is the probability that a cloudy day is followed by sunny one?

(b) What is the probability that a cloudy day is followed by two sunny ones in a row?

(c) If Friday is sunny, what is the probability that the following Sunday is sunny too?

(d) Compute $\boldsymbol{P}^2$, $\boldsymbol{P}^4$, $\boldsymbol{P}^8$, $\boldsymbol{P}^{16}$ and $\boldsymbol{P}^{32}$ by repeatedly squaring the matrices you compute. What is your conclusion? What is the practical implication regarding the weather?

202. (Continuation from 102.) Assume that the current ordering of the books is BCA. What is the probability that the ordering is ABC after two persons having used the dictionary?

203. Consider a Markov chain on the state space $\{1, 2, 3\}$ and transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

The chain starts in state 1. Compute the probability that the first three states of the chain are all different.

204. (Continuation from 118.) Compute the probability that when the server shuts down for maintenance, this is finished before a new customer arrives.

205. A simple model of DNA is that the sequence of the bases A, C, G and T forms a Markov chain. Assume this chain has transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 0.30 & 0.22 & 0.21 & 0.27 \\ 0.28 & 0.22 & 0.30 & 0.20 \\ 0.23 & 0.32 & 0.23 & 0.22 \\ 0.18 & 0.22 & 0.30 & 0.30 \end{pmatrix};$$

here the first row and column corresponds to A, etc. A subsequence that has been analysed reads ATGxxCGT, where 'xx' means that one is uncertain about these two bases. Biological considerations however suggest that these two symbols are either AC ot TG. Which of these two alternatives is the most probable?

206. A Poisson process is a Markov process on the state space $\{0, 1, 2, \ldots\}$ with transition intensities $q_{ij} = \lambda$ for $i = 0, 1, 2, \ldots$ and $j = i + 1$, and $q_{ij} = 0$ for any other $j \neq i$. In other words, from a state i the process can only jump to state $i + 1$, and it does so with intensity λ.

(a) Write down the intensity matrix $\boldsymbol{Q}$ for this model.

(b) Write down Kolmogorov's forward and backward equations.

(c) Show that both of these sets of equations are solved by

$$p_{ij}(t) = \begin{cases} e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!} & \text{for } j \geq i, \\ 0 & \text{otherwise.} \end{cases}$$

(d) Consider each transition of the Poisson process as an 'event', for example a decay from a radioactive sample or a arrival of a call to a telephone exchange. Then these events occur at constant rate. What is the probability that k events occur in a time interval $(s, s + t]$?

Time-dependent probabilities

301. (Continuation from 201.) Suppose we start observing the weather a cloudy Friday, that is, $X_0 =$ cloudy.

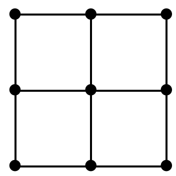
   (a) What is $\boldsymbol{p}(0)$?

   (b) Compute $\boldsymbol{p}(1)$; what is the probability that the Saturday is sunny?

   (c) What is the probability of a cloudy Sunday?

   (d) What is $\boldsymbol{p}(1)$ if $\boldsymbol{p}(0) = (1/3,\ 2/3)$?

302. A particle is placed randomly at one of the nine points below. Thereafter is does a random walk in the sense that in each step, it moves to one of the neighbouring points with equal probabilities. Hence, the particle never stays in its current position, nor moves diagonally. Compute the probability that after three steps, the particle is at the middle point.



   Hint: This problem can be solved by constructing a Markov chain with nine states, but three state are in fact enough. How should you choose them?

303. The state of a payphone, 'available' or 'occupied', is modelled as a Markov process. When the phone is available is becomes occupied with constant rate λ and when it is occupied it becomes available with rate μ. The is no queue. Finally we assume that the phone is available at time $t = 0$.

   (a) Compute the probability that the phone is available at time t.

   (b) Compute the probability that the phone is available during the whole interval $[0, t]$.

304. (continuation) Suppose that $\lambda > \mu$. After how long time is the probability that the phone is occupied larger than the probability that it is available?

305. Let $\{X(t)\}$ be a discrete Markov process on $\{0, 1, 2\}$ with model graph

$$\mathbf{0} \xrightarrow{\quad\alpha\quad} \mathbf{1} \xrightarrow{\quad\beta\quad} \mathbf{2}$$

   We assume $0 < \alpha < \beta$ and that the process starts in state 0.

   (a) Compute the time-dependent state probabilities $p_i(t) = P(X(t) = i)$ for $t \geq 0$ and $i = 0, 1$.

   (b) It is obvious that the process will eventually get absorbed in state 2. In some applications one is interested in probabilities of the kind

$$q_i(t) = P(X(t) = i \,|\, \text{absorption has not occurred at } t),$$

   where i is one of the transient states 0 and 1. Particularly interesting is whether the limit $\lim_{t\to\infty} q_i(t)$ exists for $i = 0, 1$. In this case we denote it q_i and call $\mathbf{q} = (q_0, q_1)$ a *quasi-stationary distribution*. Find out if the above process possesses a quasi-stationary distribution and, if so, compute $\mathbf{q}$.

Classification and stationarity

401. A Markov chain with states $1, \ldots, 4$ has transition probability matrix

$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix}.$$

Find out which states are recurrent and which are transient.

402. A Markov chain with states $1, \ldots, 4$ has transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Which states are recurrent, and which are transient?

(b) Compute the periods of states 1 and 2.

403. A Markov chain with states $1, \ldots, 4$ has transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 0.3 & 0 & 0 & 0.7 \\ 0.2 & 0 & 0.8 & 0 \\ 0 & 0.6 & 0 & 0.4 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Which states are recurrent and which are transient?

(b) Compute the periods of states 2 and 4, respectively.

404. A Markov chain has transition probability matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

Compute the stationary distribution $\boldsymbol{\pi}$ and find out if $\boldsymbol{p}(n) \to \boldsymbol{\pi}$ as $n \to \infty$, independently of the initial distribution.

405. A Markov chain has transition probability matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 1/3 & 2/3 & 0 \end{pmatrix}.$$

Compute the stationary distribution $\boldsymbol{\pi}$ and find out if $\boldsymbol{p}(n) \to \boldsymbol{\pi}$ as $n \to \infty$, independently of the initial distribution.

406. (a) Construct an irreducible Markov chain with 3 states and period 2. Give your answer as a model graph.

(b) Construct a Markov chain with 3 states and no loops ($p_{ii} = 0$ for each state i), having a unique stationary distribution $\boldsymbol{\pi}$ and the property that $\boldsymbol{p}(n) \to \boldsymbol{\pi}$ då $n \to \infty$ irrespective of the initial distribution. Give your answer as a model graph and motivate carefully why your Markov chain has the requested properties.

407. The concentration of a certain chemical compound outside a process industry is measured once per day and is then labelled as either normal (N) or high (H). The sequence of labels can be modelled by a discrete Markov chain. The probability that a normal value is followed by a high one is 0.1, and the probability that a high value is followed by a normal one is 0.95.

    (a) Compute the probability that a high concentration is measured any given day.

    (b) Compute the expected number of days during a period of 30 days, at which a high value is measured, but the following day shows a normal one.

408. (Continuation from 103.) Under stationarity, find out the fraction vehicles that are lorries. Do the computation for both models, (a) and (b).

409. An company signs each year a contract for delivery of a certain product with one of three possible subcontractors. The sequence of chosen subcontractors forms a Markov chain with transition probability matrix

$$\begin{pmatrix} 2/3 & 1/3 & 0 \\ 2/3 & 1/4 & 1/12 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

    Compute the probability that, after a long time, the same subcontractor is chosen two years in a row.

410. (Continuation from 107.) Compute the probability that, after the show has been going on for a long time, any given caller wins three CDs on more.

    When you have solved this problem using Markov chains, convince yourself that it can be solved without using Markov chains whatsoever—and do it!

411. A building has five floors, including the ground floor, and one elevator. When the elevator is at the ground floor its next stop is any of the other floors, with equal probabilities. When the elevator is not at the ground floor its next stop is the ground floor with probability 0.7, and any of the other floors—except the current one—with equal probabilities.

    (a) Compute the probability that if the elevator currently is at the ground floor, it will be there also three stops later.

    (b) Compute the probability that after a long time, the elevator is at the ground floor.

412. A small elevator can carry at most two persons. The elevator goes from the ground floor to the first floor, and back again. The duration of such a round trip has an exponential distribution with mean $1/\mu$. People arrive to the elevator at the ground floor according to a Poisson process of rate λ. In case the elevator is available and there is just one person waiting at the ground floor, this person does not wait for the next one but takes the elevator to the first floor at once. Compute the stationary probability of n persons waiting at the ground floor.

    Hint: Define the states $0'$: elevator available, 0: elevator occupied but nobody is waiting, and n: n persons are waiting (the elevator is then occupied).

    Put $\pi_n = \pi_0 \alpha^n$ for $n \geq 0$ and solve for α. When does a stationary solution exist at all?

413. A stochastic process in discrete time and with discrete state space is said to be a Markov chain of order m if

$$\begin{aligned} P(X_{n+1} = i_{n+1} \,|\, X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) \\ = \quad P(X_{n+1} = i_{n+1} \,|\, X_n = i_n, \ldots, X_{n-m+1} = i_{n-m+1}) \end{aligned}$$

    for all $n \geq m-1$ and $i_0, i_1, \ldots, i_{n+1}$ in the state space. In other words, the next value X_{n+1} depends (stochastically) on the m previous values $X_n, \ldots, X_{n-m+1}$. An ordinary Markov chain is thus a Markov chain of order 1.

    Consider a Markov chain of order 2 with transition probabilities $p_{i,j;k} = P(X_{n+1} = k \,|\, X_n = i, X_{n-1} = j)$ given by the following table:

i	j	k	$p_{ij;k}$
0	0	0	0.4
0	0	1	0.6
0	1	0	0.8
0	1	1	0.2
1	0	0	0.5
1	0	1	0.5
1	1	0	0.3
1	1	1	0.7

Define the random variable Z_k as the couple $Z_k = (X_k, X_{k-1})$. Then $\{Z_k\}$ is an ordinary (order 1) Markov chain with state space $\{(0,0), (0,1), (1,0), (1,1)\}$ (you do not need to prove this).

(a) What is the transition probability matrix $\boldsymbol{P}$ of $\{Z_k\}$?

(b) Compute the stationary probability $P(X_k = 0)$.

414. A Markov process has intensity matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{pmatrix}.$$

Compute the stationary distribution $\boldsymbol{\pi}$. Is it unique? Does $\boldsymbol{p}(t) \to \boldsymbol{\pi}$ as $t \to \infty$, whatever $\boldsymbol{p}(0)$?

415. A birth-and-death process on the state space $\{0, 1, 2, 3, \ldots\}$ has birth intensities $\lambda_n = \lambda/\sqrt{n+1}$ and death intensities $\mu_n = \mu\sqrt{n}$. Compute the stationary distribution.

416. (Continuation from 111.) What fraction of time is the machine down?

417. (Continuation from 114.) Compute the stationary probability that both servers are busy.

418. (Continuation from 115.) Compute the probability that, after a long time, there are no customers in the system.

419. The service in a small shop with one clerk can be modelled as an M/M/1 queueing system with infinite waiting space and load $\rho = \lambda/\mu = 0.9$. The shop itself is limited, however, so that only a certain number of waiting customers can wait inside—possibly remaining customers must queue outside. How large does the shop need to be, or, in order words, how many customers does it need to fit, in order to make the probability that there are customers waiting outside no larger than 5%? Assume stationarity.

Hint: Look at Example 4.8 in the lecture notes.

420. Four customers each move forth and back between two service stations A and B. At A, service times have an exponential distribution with mean $1/a$, and at B they have an exponential distribution with mean $1/b$. Each service station has one server, and transportation times between the stations are assumed neglectable. Hence, each customer is either in service or waiting in a queue.

Let $X(t)$ be the number of customers at station A, either in service or waiting, at time t. Compute the stationary distribution of this Markov process.

421. (Continuation from 119.) Compute the probability that the storage is full, assuming $\lambda_A \neq \lambda_B$. Express your answer in terms of $\rho = \lambda_A/\lambda_B$.

422. (Continuation from 120.) Compute the stationary probability that the storage is empty. This probability equals the probability that the storage is empty when a unit is requested by the production (you do not need to prove that!).

423. Consider a particular component in a system. The component is critical in the sense that if it fails, so does the system. The life lengths of these component have an exponential distribution with mean $1/\lambda$. The system is always under surveillance, except for a period right after a new critical component has been installed; these periods of non-surveillance have lengths which have an exponential distribution with mean $1/\mu$. When the critical component fails it is immediately replaced by a new one if the system is under surveillance, otherwise the replacement is made when the surveillance is restarted. All component life lengths and periods of non-surveillance are independent.

     Compute the stationary probability that the system has a working critical component.

424. (Continuation from 116.) Compute the stationary probability that the server is turned on.

425. (Continuation from 117.) Assuming that the system admits a stationary distribution, compute the stationary probability that server A is busy.

426. (Continuation from 118.)

     (a) Show that the stationary distribution $\boldsymbol{\pi}$ satisfies $\pi_{(k,D)} = \{\lambda/(\lambda+\delta)\}^k \pi_{(0,D)}$ for $k \geq 1$.

     (b) Show that the stationary distribution $\boldsymbol{\pi}$ satisfies

$$\pi_{(k,U)} = \sum_{i=0}^{k-1} \left(\frac{\lambda}{\mu}\right)^{k-i} \pi_{(i,D)} + \left(\frac{\lambda}{\mu}\right)^{k} \pi_{(0,U)}$$

     for $k \geq 1$.

     Hint: Make a well chosen cut through the model graph. These results may be used to solve for the stationary probabilities.

427. A transition probability matrix of a Markov chain always has row sums equal to one. If, in addition, all column sums are one, the matrix is called *doubly stochastic*. Prove that if $\{X_n\}_{n=0}^{\infty}$ is an irreducible Markov chain with finite state space, $\{1, 2, \ldots, r\}$ say, and doubly stochastic transition probability matrix, then its stationary distribution $\boldsymbol{\pi}$ is uniform on the state space, that is, $\pi_i = 1/r$ for all $i = 1, 2, \ldots, r$.

428. A binary information source is modelled as follows. Let $\{X_n\}_{n=0}^{\infty}$ be a Markov chain with state space $\{0, 1, 2\}$ and transition probability matrix

$$\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.7 & 0 \end{pmatrix}.$$

     Moreover, let Y_n be information bit n. Assume that $Y_n = 0$ if $X_n = 0$, and that $Y_n = 1$ if $X_n = 1$ or 2. Thus, we can observe Y_n, but not X_n. Suppose also that $\{X_n\}$ is stationary.

     (a) For purposes of coding, e.g., it may be interesting to estimate the unobserved state X_n. The more Y-values that are known, the better the estimate. Ideally we would like to compute, for example, $P(X_n = 1 \mid Y_n, Y_{n-1}, \ldots)$, and this is in fact quite possible. We shall, however, for now settle for something simpler. Compute $P(X_n = 1 \mid Y_n = 1)$.

     (b) Predictions are also of interest. Compute $P(Y_{n+1} = 0 \mid Y_n = 1)$.

429. For a Markov chain, the global balance equations $\pi_j = \sum_i \pi_i p_{ij}$ for all j characterise the stationary distribution(s). The equations

$$\pi_i p_{ij} = \pi_j p_{ji} \quad \text{for all } i \text{ and } j$$

     are called the *local balance equations*.

     (a) Prove that if the local balance equations are satisfied, so are the global ones. Hence, if $\boldsymbol{\pi}$ is a distribution that satisfies the local balance equations, it is a stationary distribution.
     Hint: Note that, for all j, $\sum_i p_{ji} = 1$.

(b) Find an irreducible Markov chain with a stationary distribution that does not satisfy the local balance equations.

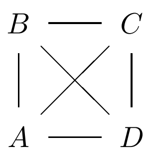
(c) Find out the right way of formulating local balance equations in continuous time, that is, for Markov processes. Show that, again, local balance implies global balance.

Our conclusion is that local balance equations may be used to find stationary solutions—and in fact they typically yield simpler equations than what global balance does—but it is a method that does not always work.

Absorption

501. (Continuation from 302.) If the particle starts in a corner, what is the average number of steps it needs to take before reaching the middle point?

502. A particle is placed at corner A in the picture below and then moves between the corners by taking steps in discrete time. In each step the particle chooses, with equal probabilities, one of the corners where it currently not located. All steps are made independently of each other. Compute the average number of steps the particle needs to take in order to visit all corners.



503. A spider chasing a fly moves between two locations, 1 and 2, as a Markov chain with transition probability matrix

$$\boldsymbol{P}_S = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}.$$

The spider starts at location 1. The fly starts at location 2 and moves as a Markov chain with transition probability matrix

$$\boldsymbol{P}_F = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}.$$

The spider catches the fly when they meet at the same location.

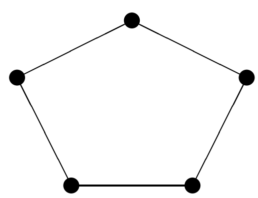
(a) Show that the chase can be described by a Markov chain with three states.

(b) Compute the expected length of the chase.

504. Consider a stream of 0's and 1's, independent and with each symbol being 0 with probability $1/2$. Compute the average number of symbols we need to record, on the average, until the sequence 000 has appeared for the first time.

For example, in the sequence 010111001000 the sequence 000 appears for the first time after 12 symbols.

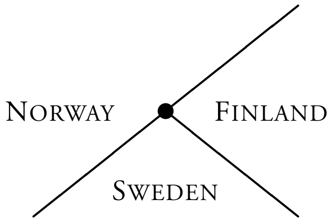
505. Five nodes are connected in a ring:



Two particles are placed at two adjacent nodes, and then move according to two independent random walks on this graph; at each step each particle moves either clockwise or counter-clockwise with equal probabilities, and independently of the other particle.

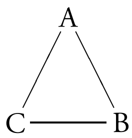
Compute the average time it takes before the particles meet, i.e. are at the same node after a completed step.

506. Three hikers agree to meet at Treriksröset (the northernmost point of Sweden), where Sweden, Norway and Finland meet, to celebrate Midsummer.



They all arrive at time $n = 0$, but, unfortunately, all in different countries. At time points $n = 1, 2, 3, \ldots$ each of them moves to a randomly chosen different country (that is, different from where he/she currently is), independently of each other. How long does it take, on the average, before all three hikers are in the same country and can start the celebration? (They bring herring, *snaps* and strawberries, respectively, and the party cannot start until everything is in place!)

507. Consider a particle that does a discrete time random walk on the triangle below.



Starting at corner A, in each step the particle moves to one of the two corners where it is currently not, with equal probabilities. Compute the expected time until the particle has, for the first time, completed a full turn around the triangle (in either direction).

For example, in the path ABACABC a full turn is completed after 6 steps (clockwise), and in the path ACBCACBCBA a full turn is completed after 9 steps (anticlockwise). Draw pictures to see what is going on!

508. Anne and Birger play a ball game. If the one who serves wins the ball, he/she gets one point. If the server loses the ball, nobody gets a point and the other player serves the next ball. A ball is won by the server with probability 2/3. The outcome of different balls, given the servers, are independent.

Anne starts to serve. Compute the average number of points she gets before Birger gets his first one.

509. A process industry sometimes exceeds the allowed emission levels of chemical compounds. When the process is in a normal state (with legal emissions) it moves into an abnormal state (with illegal emissions) with the intensity 0.02 day^{-1}. When the process in the abnormal state it is corrected; this takes an independent exponentially distrubuted random time to accomplish, with mean 1/2 day. The industry is monitored by authorities at time-points from an independent Poisson process with rate 0.03 day^{-1}. At time $t = 0$ the process is in the normal state. Compute the expected time until the process is monitored when it is in the abnormal state.

510. Consider the following model for patients that have received treatment for a specific form of cancer. The patient may be in each of the following four stages (states):

0 – Initial state of being under treatment;

1 – State of being deceased immediately following treatment for cancer (death from cancer or operative death);

2 – State of recovery in which the patient is not under treatment but under observation;

3 – State of being deceased by causes not connected to cancer.

Only transitions $0 \to 1$, $0 \to 2$, $2 \to 0$ and $2 \to 3$ are possible, and the respective intensities are denoted by q_{ij} for appropriate i and j. Compute the average time until death (by cancer or other cause) for a patient that is in state 0.

511. Men and women arrive at a store according to two independent Poisson processes with intensities 4 and 5, respectively. That is, men and women arrive with constant intensities 4 and 5, respectively, independently of each other. Each customer visits the store for a time that is random and with an exponential distribution; the mean of this exponential distribution is 3 and 5 for men and women, respectively. All random variables are independent.

When the store opens it is empty and the are no customers waiting outside. Compute the average time until, for the first time, there are two customers in the store.

Hint: Think carefully about how to choose states.

512. A Markov chain with state space $\{1, 2, 3, 4\}$ has transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & a_2 & a_3 & a_4 \\ b_1 & 0 & b_3 & b_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where, obviously, $a_2 + a_3 + a_4 = b_1 + b_3 + b_4 = 1$.

States 3 and 4 are both absorbing and states 1 and 2 are transient. Thus the chain will eventually be absorbed in state 3 or 4. Compute the probability that if the chain starts in state 1, it gets absorbed in state 3.

Inference

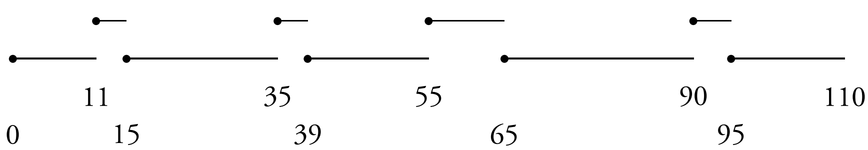
601. Novels have lots of uses—other than reading. For instance, Andrej Markov used the 20,000 first letters of Aleksandr Pusjkin's novel *Eugene Onegin* for inventing Markov chains.

If one considers 'vowels' and 'consonants' as two different states of a Markov chain one may (and Markov did!) write down the following transition probability matrix, describing how vowels and consonants follow upon another in Russian:

$$\hat{\boldsymbol{P}} = \begin{pmatrix} 0.13 & 0.87 \\ 0.66 & 0.34 \end{pmatrix}.$$

Compute approximate 95% confidence intervals for the transition probabilities between vowels and consonants, respectively.

602. A switch toggles, apparently at random, between the states 0 and 1. One has observed the following time points of switches:



The final observation is censored. Assume that the position of the switch follows a Markov process with states 0 and 1, and intensity matrix

$$\boldsymbol{Q} = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

Compute the maximum likelihood estimates of α and β.

The Poisson process

701. Measurements in a telephone exchange show that, during business hours, a telephone call in progress ends with constant intensity $\lambda = 0.5$ per minute.

    (a) What is the distribution of the duration of a call?

    (b) What is the probability that a call lasts no longer than one minute?

    (c) What is the probability that a call that has lasted for one minute already, will be finished during the next minute?

702. A certain type of electrical components have a constant failure intensity λ. A box contains a number of such components, all of age t, and one picks a component from the box that is still working. Compute the total expected life length of this component.

703. At a construction site, accidents occur according to a Poisson process of intensity 0.1 per working day. What is the distribution of the number of accidents during a working week? What is the probability that two or more accidents occur during one day? How realistic is the Poisson process assumption?

704. Consider a Poisson process in which four events occurred in the interval $(0, 4]$. Compute the probability that two of these occurred in the interval $(0, 1]$ and the remaining ones in the interval $(1, 2]$.

705. In a road traffic survey the number of cars passing a marker at a road was counted. The streams of cars in the two directions were a priori modelled as independent Poisson processes of intensities 2 and 3 per minute, respectively. It was decided to stop the counting once 400 cars had passed. Let T_{400} be that (random) time point and compute, using appropriate approximations, a time a such that $P(T_{400} \leq a) = 0.90$.

706. Consider a service system to which two streams of customers arrive: one Poisson process with intensity 2 s^{-1} in which each event corresponds to the arrival of one customer, and one Poisson process with intensity 1 s^{-1} in which each event corresponds to the arrival of *two* customers. The two Poisson processes are independent of each other.

    Compute the probability that during 1 s, exactly four customers arrive.

707. An operator sits down at a switchboard. Call arrivals may be modelled as a Poisson process such that, on the average, there is one call every second minute. What is the probability that the first call does not arrive until after two minutes or later?

708. (continuation) The operator leaves the switchboard for three minutes. When she returns she finds a flashing light, indicating that exactly one call is waiting. What is the distribution of the time the light has been flashing?

709. (continuation) The operator leaves the switchboard for one minute, and when she returns a display shows that two calls are waiting for attention. Compute the probability that both calls have been waiting for more than 40 s.

710. Telephone calls arrive to a switchboard as a Poisson process with rate λ. Given that at least one call arrives in the time interval $(0, t]$, what is the probability that the first call arrived during the first half of this interval?

711. Consider a Poisson process with intensity λ. Given that at least one event occurs in the interval $[0, T]$, compute the conditional probability that the first event occurred before time t $(0 < t < T)$.

712. A switch in a certain electrical system can be either 'off' on 'on', and it changes its state each time it receives an electric pulse. At time $t = 0$ the switch is off, and after that it receives pulses according to a Poisson process with intensity 0.5 s^{-1}.

    a) Compute the probability that the switch is on at $t = 2.5$ s.

    b) Compute the probability that the switch is on at $t = 2.5$ s and remains in this state until $t = 5$ s.

713. Consider a Poisson process with intensity 2. Compute (a good approximation of) the probability that the total number of events in the time interval $(0, 2]$ is exactly twice the number of events during the first half of this interval.

714. Fires in passenger trains are assumed to occur as a Poisson process. During the period January 1 1986 until December 31 1994, there were 67 fires in passenger trains in Sweden. During the same period, the total distance run by passenger trains in Sweden was 544×10^6 km. Estimate the intensity of fires (per train km) and construct a two-sided confidence interval with approximate level 95%.

715. The traffic across a bridge is described by two independent Poisson processes, one for each direction. The two intensities are called λ and μ (unit: vehicles per minute), and a previous survey has shown that $\lambda + \mu \approx 20$. Now the interest is in the two separate intensities and one wants to determine $\lambda - \mu$ with a precision of 0.5 with 95% confidence. For how long must the traffic be counted?

716. Detectors such as Geiger counters can measure decay from a radioactive sample. Such radioactive decay is well-described by a Poisson process. The background radiation, which is always present, is also measured.

Assume that one measures on a sample plus background for a time t_s and the background only for a time t_b (unit: s).

(a) Write down a statistical model that describes the measurement set-up, and write down how one can estimate the decay intensity λ_s from the sample (denote the background intensity by λ_b).

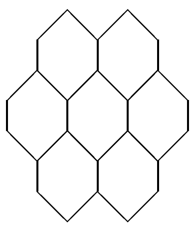
(b) It is desirable that the variance $V(\hat{\lambda}_s)$ is small. Assuming a total time t for the measurements, how shall it be split into t_s and t_b in order to minimise this variance?

(c) Construct a confidence interval for λ_s.

717. A thunderstorm passes over a village during one hour. During this hour lightnings hit the ground according to a (space-time) Poisson process with intensity 30 hits per km^2 and hour.

Consider a house of area 200 m^2. What is the expected number of hits in this house during one hour? What is the probability that the house is not hit at all?

718. A village is covered by cells for mobile telephony according to the following picture:



Calls arrive as a space-time Poisson process of intensity 3 calls per cell and minute.

a) Compute the probability that at least 5 calls arrive in the middle cell during a given minute.

b) Compute the probability that at least 5 calls arrive in at least one of the cells during the given minute.

719. In the rectangular country Snow, 2000 km tall and 500 km wide, a ski race is arranged each year in early March. The participants of the race are distributed over the country as a spatial Poisson process with intensity

$$\lambda(x, y) = 2 \times 10^{-4} y^{3/5} \text{ (km)}^{-2},$$

where x and y are the horizontal and vertical distances, respectively, from the south west corner of Snow.

(a) Compute the expected number of participants in the race.

(b) Compute the probability that the number of participants from the northern half of the country is more than double the number of participants from the southern half.

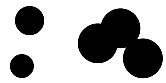
Hint: What approximation can be used for a Poisson distribution with large mean?

720. A muffin with raisins may be viewed as part of a three-dimensional Poisson process, where the raisins are the points.

(a) How many raisins does a muffin need to contain, on the average, in order to make the probability that a muffin contains no raisins at most 0.05?

(b) What objections could be raised against the Poisson process model in this case?

721. Consider the following model. Points are distributed in the plane $\mathbf{R}^2$ according to a Poisson process with rate λ (per unit area). Around every Poisson point a circular disc is placed, with radius drawn from a uniform distribution on $(0, a)$, where $a > 0$ is a parameter. The radii of different discs are drawn independently, and independent of the Poisson process of centre points. Discs may overlap each other. The following picture illustrates the model.



A model of this type is called a *germ-grain model*. A question of interest is what fraction of the plane that is covered with discs, on average. This expected fraction equals the probability that any given point in the plane is covered by (at least) one disc. Compute this probability.

Hint: You may be helped by considering the triplets (x, y, r), where (x, y) is the coordinates of a Poisson point and r is the radius of the disc centred there. The process of such triplets (x, y, r) is a Poisson process on the set $\mathbf{R}^2 \times (0, \infty)$, with intensity $\lambda f(r)$ where $f(r)$ is the probability density function of the uniform distribution above.

722. Consider a Poisson process in the three-dimensional space $\{(x, y, t) : (x, y) \in \mathbf{R}^2, t \geq 0\}$, where the first two coordinates (x, y) is a position in the plane $\mathbf{R}^2$ and the third coordinate t is a time point. The intensity of the process is λ (unit: $\mathrm{m}^{-2}\mathrm{s}^{-1}$). If (x, y, t) is a point of this Poisson process, a disc starts growing from position (x, y) at time t, with radial speed v (unit s^{-1}). This implies that at time $t + u$ $(u > 0)$ there is a disc centred at (x, y) and with radius uv. As time goes by, more and more such centre points will appear, and around each of them a disc starts growing. The picture of the previous exercise gives a snap shot of what this might look like.

The discs are different in size as they start to grow at different time points, and they may, as indicated by the figure, overlap each other.

This model is called the *Johnson-Mehl model*. One question of interest is to compute what fraction of the plane $\mathbf{R}^2$ that is covered, on average, by discs at time t. This expected fraction equals the probability that any given point of $\mathbf{R}^2$ is covered by (at least) one disc at time t. Compute this probability.

723. In a liquid, the number of bacteria in a certain volume unit has a Poisson distribution with mean m; the numbers of bacteria in disjoint parts of the liquid are independent. In a study of 100 volume units one found a total of 17 bacteria.

(a) Compute an estimate of m and a confidence interval with approximate level 95%.

(b) Compute a confidence interval with approximate level 95% for the probability that a volume unit of the liquid does not contain any bacteria.

724. Consider an area A of the earth in which earthquakes sometimes occur. We model the occurrences of earthquakes over space and time as a Poisson process of rate $\lambda(x, y)$ (unit: $(\mathrm{km})^{-2}\mathrm{s}^{-1}$). The intensity is thus independent of time t, but does depend on the spatial coordinates $(x, y) \in A$; earthquakes are more common in some regions of A than in other.

(a) Let $t = 0$ correspond to today. What is the distribution of the waiting time until the first earthquake in A?

(b) Split A into two disjoint subregions B and C. That is, $A = B \cup C$. What is the probability that there is an earthquake in B before there is one in C?

Hint: If U and V are two independent exponential random variables with different means, what is the probability $P(U < V)$?

Renewal processes

Exercises 801–806 are (with some modifications) from S.M. Ross: *An Introduction to Probability Models*, 3rd ed., Academic Press, 1995.

801. Beverly has a radio which works on a single battery. As soon as the battery in use fails, Beverly immediately replaces it with a new one. If the lifetime of a battery (in hours) is distributed uniformly over the interval (30,60), then

(a) at what rate does Beverly have to change batteries?

(b) when Beverly has used her radio for a long time, what is the distribution of the age of her battery?

802. Is it true that

(a) $N(t) < n$ if and only if $S_n > t$?

(b) $N(t) \leq n$ if and only if $S_n \geq t$?

(c) $N(t) > n$ if and only if $S_n < t$?

803. Mr. Smith works on a temporary basis. The mean length of each job he gets is three months. If the amount of time he spends between jobs is exponentially distributed with mean two weeks, then at what rate does Mr. Smith get new jobs?

804. Suppose that the inter-event time distribution for a discrete-time renewal process is Poisson with mean a. That is, suppose

$$P(Y_n = k) = \frac{a^k}{k!} e^{-a} \quad \text{for } k = 0, 1, 2, \ldots.$$

Note that we here temporarily abandon the assumption $f_0 = 0$.

(a) Find the distribution of S_n.

(b) Calculate $P(N_n = k)$.

805. A machine in use is replaced by a new machine either when it fails or when it reaches the age of T years. If the lifetimes of successive machines are independent with a common distribution F having density f, show that

(a) the long-run rate at which machines are replaced equals

$$\left[\int_0^T x f(x)\, dx + T(1 - F(T)) \right]^{-1}.$$

(b) the long-run rate at which machines in use fail equals

$$\frac{F(T)}{\displaystyle\int_0^T x f(x)\, dx + T(1 - F(T))}.$$

806. Suppose that potential customers arrive at a single-server bank in accordance with a Poisson process having rate λ. However, suppose that the potential customer only will enter the bank if the server is free when he/she arrives. That is, if there is already a customer in the bank, then our arrivee, rather than entering the bank, will go home. If we assume that the amount of time spent in the bank by an entering customer is a random with mean a, then

    (a) what is the rate at which customers enter the bank?

    (b) what proportion of customers actually enter the bank?

    (c) do the time points when customers enter the bank constitute a (possibly delayed) renewal process?

    (d) do the time points when customers leave the bank constitute a (possibly delayed) renewal process?

807. Prove the part of Theorem 5.4 of the notes that concerns the backward recurrence time for a discrete time renewal process.

    You can either first find the transition probabilities of $\{A_n\}$, or try to express the backward recurrence time in terms of forward recurrece times (or, even better, do both!).

808. Derive the renewal equation for the distribution of the backward recurrence time stated in Example 5.3 of the notes. Then use the key renewal theorem to assess that the limiting distribution of $B(t)$ is as stated in Theorem 5.4. Remember to check that the conditions of the key renewal theorem are satisfied!

809. Let $\{Y_n'\}_{n=1}^{\infty}$ and $\{Y_n''\}_{n=1}^{\infty}$ be two independent sequences of i.i.d. positive random variables with continuous distributions. We call these sequence the *primary sequence* and the *secondary sequence*, respectively. To make things more concrete we will think of these sequences as *times to failure* and *repair times* in a reliability context. In this context, a system alternates between being up and being down (under repair). This is modelled by constructing the sequence $(Y_1', Y_1'', Y_2', Y_2'', Y_3', \ldots)$, alternating between up and down intervals. Likewise, put $S_1' = Y_1'$, $S_1'' = Y_1' + Y_1''$, $S_2' = Y_1' + Y_1'' + Y_2'$, $S_2'' = Y_1' + Y_1'' + Y_2' + Y_2''$, etc. Thus S_n' is the time of the n-th failure and S_n'' is the time of the n-th completed repair. The sequence $(S_1', S_1'', S_2', S_2'', \ldots)$ is often called an *alternating renewal process*. We notice that with $Y_k = Y_k' + Y_k''$ and $S_n = \sum_1^n Y_k$ as usual, $\{S_n\}$ is a (pure) renewal process. Although described in reliability terms here, alternating renewal processes are useful in many other contexts as well.

    It is obviously of interest to compute the probability that the system is up at some time t. Thus write $X(t) = 1$ if the system is up at t, and $X(t) = 0$ otherwise. Then

    $$\begin{aligned} P(X(t) = 1) &= P(X(t) = 1, Y_1' > t) + P(X(t) = 1, Y_1' \le t) \\ &= P(Y_1' > t) + P(X(t) = 1, Y_1' \le t), \end{aligned}$$

    since the event $Y_1' > t$ implies that $X(t) = 1$. Now condition on the time Y_1 of the first event in the renewal process to show that with $Z(t) = P(X(t) = 1)$,

    $$Z(t) = P(Y_1' > t) + \int_0^t Z(t-x) f(x)\, dx,$$

    where as usual f is the density function of the Y_k.

    Use this renewal equation to derive an expression for the limit of $P(X(t) = 1)$ as $t \to \infty$.

810. Argue that in a Poisson process, the forward and backward recurrence times $A(t)$ and $B(t)$ are independent random variables. What are their distributions? Define the *current lifetime* $C(t) = A(t) + B(t)$, i.e. the length of the interval in which the time point t falls. Argue (just using properties of the Poisson process you have given in this exercise), that for large t, $C(t)$ has approximately the density

    $$f_{C(t)}(x) = \lambda x e^{-\lambda x} \quad \text{for } x \ge 0,$$

    where λ is the rate of the process.

811. Let $C(t) = A(t) + B(t)$ be the current lifetime as above for a continuous time renewal process and let $Z(t) = P(C(t) > x)$. Show that $Z(t)$ satisfies the renewal equation

$$Z(t) = 1 - F(\max(t, x)) + \int_0^t Z(t-y) f(y)\, dy.$$

Show that under appropriate conditions, $Z(t)$ tends to $(1/\mu)\int_x^\infty u f(u)\, du$ as $t \to \infty$. (Hint: after applying the key renewal theorem, write $1 - F(x) = \int_x^\infty f(u)\, du$ and change order of integration.)

Use this result to recover the second part of Exercise 810.

812. Consider a continuous time renwewal process and let $Z(t) = P(A(t) > x, B(t) > y)$. Show that $Z(t)$ satisfies the renewal equation

$$Z(t) = (1 - F(t + x)) I\{t > y\} + \int_0^t Z(t-u) f(u)\, du,$$

where $I\{t > y\}$ is the *indicator function* of the event $\{t > y\}$, that is

$$I\{t > y\} = \begin{cases} 1 & \text{if } t > y \\ 0 & \text{if } t \le y \end{cases}$$

Show that

$$Z(t) \to \int_{x+y}^\infty (1 - F(u))\, du \quad \text{as } t \to \infty.$$

Use this result to recover the first part of Exercise 810.

Regenerative processes

901. Packages arrive at a mailing depot according to a Poisson process with rate λ. Trucks, picking up all waiting packages, arrive according to a renewal process with interarrival time density f. Let $X(t)$ denote the number of packages waiting to be picked up at time t.

   (a) Is $\{X(t)\}$ regenerative? Why or why not?

   (b) Derive an expression for $\lim_{t\to\infty} P(X(t) = k)$.

902. Show that the current life time process $C(t)$ in Exercise 811 is regenerative with respect to the underlying renewal process. Derive the asymptotic distribution of this process (the result given in Exercise 811) using regenerative process theory.

903. Consider an alternating renewal process as in Example 809. Show that the process $\{X(t)\}$ defined there is regenerative with respect to the renewal process $\{S_n\}$ of time points when repairs are completed.

   Derive the asymptotic distribution of $X(t)$ using regenerative process theory.

   Note that it may make sense to assume that the amount of repair time required depends on the time the system was up, and that such an assumption does not invalidate the arguments of this exercise.

904. Consider a G/G/1 queue. A *busy period* is the period starting with an arrival to an empty queue and ending with a customer leaving an empty system behind. An *idle period* is the period until the next customer arrives (to an empty system). A *regeneration cycle* (see Example 6.1 in the notes) is thus the sum of a busy period and an idle period.

   Now look at an M/G/1 queue. What is the distribution and expected length of an idle period? (Hint: the Poisson process and its poor memory.)

   Can you express the stationary probability of a busy server, i.e. $P_e(Q(\cdot) \ge 1)$, in terms of expected lengths of busy and idle periods? (Yes, you can! Hint: look at Exercise 809.) This probability is, for any G/G/1 system, $\rho = \lambda d$, where λ is the arrival rate and d is the mean service time. Use this to compute the expected length of a busy period in an M/G/1 system.